

1. Introduction

Typical basis systems for the modeling of geomagnetic quantities are the well-known spherical harmonics. Each degree of such a basis function corresponds to one fixed frequency, thus, allowing an interpretation as multipoles. However, the uncertainty principles states that frequency localization and space localization are mutually exclusive, such that it is desirable to additionally have a basis system resolving the space evolution of the modeled quantities. Some approaches towards this have been made, for example, by spherical cap harmonic analysis and wavelets as superpositions of vector spherical harmonics (see [1],[2]). Both approaches are still based on frequency oriented basis systems and therefore somewhat artificially localized. A new approach based on the mathematical properties of Green's function with respect to the Beltrami operator delivers expansions of the modeled quantities in terms of purely space localizing wavelets (see [3]), which can be applied to many geomagnetic problems where the typical spherical differential operators are of importance.

2. Green's Function for the Beltrami Operator

By Green's function with respect to the Beltrami operator we denote the function

$$G(\Delta^*; t) = \frac{1}{4\pi} \ln(1-t) + \frac{1}{4\pi} (1 - \ln 2), \quad -1 \leq t < 1.$$

Integral formulas as derived in [4] allow a solution of the Beltrami differential equation

$$\Delta^* F = H$$

by the convolution

$$F(\xi) = \int_{\Omega} G(\Delta^*; \xi \cdot \eta) H(\eta) d\omega(\eta), \quad \xi \in \Omega.$$

A regularization of Green's function

$$G^{\rho}(\Delta^*; t) = \begin{cases} \frac{1}{4\pi} \ln(1-t) + \frac{1}{4\pi} (1 - \ln 2), & -1 \leq t \leq 1-\rho, \\ \frac{1}{4\pi\rho} (1-t) + \frac{1}{4\pi} (\ln \rho - \ln 2), & 1-\rho < t \leq 1, \end{cases}$$

allows to define scaling functions

$$\varphi^0(\xi, \eta) = G^{\rho_0}(\Delta^*; \xi \cdot \eta)$$

and locally supported wavelets

$$\psi^j(\xi, \eta) = G^{\rho_{j+1}}(\Delta^*; \xi \cdot \eta) - G^{\rho_j}(\Delta^*; \xi \cdot \eta).$$

This delivers an expansion

$$F(\xi) = \int_{\Omega} \varphi^0(\xi, \eta) H(\eta) d\omega(\eta) + \sum_{j=0}^{\infty} \int_{\Omega} \psi^j(\xi, \eta) H(\eta) d\omega(\eta),$$

where the wavelet convolutions represent the local information added at each step of the procedure. (For the actual modeling of geomagnetic quantities the scaling functions and wavelets have to be adapted to the treated problem by the same procedure as in the Beltrami equation case)

3. Mie-/ Helmholtz Representation

As summarized in [5], every continuously differentiable vector field can be decomposed into toroidal and poloidal parts by the Mie decomposition

$$\mathbf{F} = \nabla \times \mathbf{L}P_{\mathbf{F}} + \mathbf{L}Q_{\mathbf{F}},$$

and into orthogonal parts by the Helmholtz decomposition

$$\mathbf{F}(\xi) = \xi F_1(\xi) + \nabla^* F_2(\xi) + \mathbf{L}^* F_3(\xi).$$

A combination of both representations delivers

$$\mathbf{F}(\mathbf{r}\xi) = \xi \frac{\Delta^* P_{\mathbf{F}}(\mathbf{r}\xi)}{r} + \nabla^* \frac{\partial}{\partial r} r P_{\mathbf{F}}(\mathbf{r}\xi) + \mathbf{L}^* Q_{\mathbf{F}}(\xi).$$

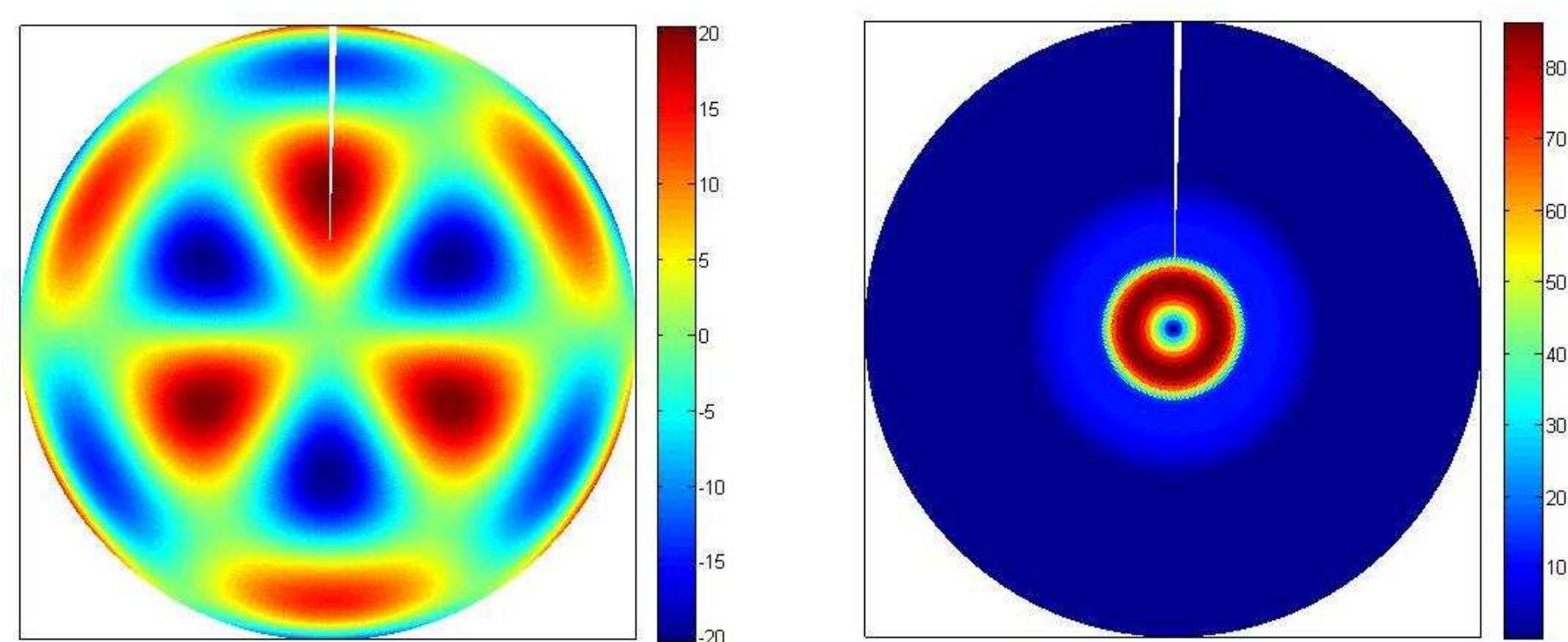


Fig. 1: Spherical Harmonic Y_{63} of degree 6 (left) and wavelet kernel at scale 4 (right).

4. Application to Geomagnetic Modeling

4a. Ionospheric Currents

The pre-Maxwell equations deliver a connection between the Mie scalars for the magnetic field and the corresponding currents,

$$Q_{\mathbf{B}} = \frac{1}{\mu_0} P_{\mathbf{J}},$$

$$\Delta P_{\mathbf{B}} = -\frac{1}{\mu_0} Q_{\mathbf{J}},$$

that has also been applied in [6] to the determination of ionospheric F-region currents. In our approach a combination with the Helmholtz decomposition delivers the following wavelet reconstruction for radial currents

$$J_{\text{rad}}(\mathbf{r}\xi) = \frac{1}{\mu_0 r} \int_{\Omega} \varphi^0(\xi, \eta) \cdot \mathbf{B}(\eta) d\omega(\eta) + \sum_{j=0}^{\infty} \frac{1}{\mu_0 r} \int_{\Omega} \psi^j(\xi, \eta) \cdot \mathbf{B}(\eta) d\omega(\eta),$$

with the scaling kernel

$$\varphi^j(\xi, \eta) = L_{\eta}^* \Delta_{\xi}^* G^{\rho_j}(\Delta^*; \xi \cdot \eta),$$

and wavelets as the differences of such kernels, therefore satisfying the desired property of local support (a sample reconstruction of an analytic current from [7] up to scale 9 is presented in Fig. 2).

4b. Upward Continuation

As is the case for the *Swarm* mission, magnetic data that is available at different altitudes needs to be upward or downward continued to the same altitude to be comparable. This is only possible over source free regions, such that a split up of the measured quantities into parts produced by sources from the interior and sources from the exterior becomes necessary (for theoretical considerations, see [5]), for a wavelet implementation, [2]). The magnetic field can then be represented as a potential field

$$\mathbf{B} = \nabla U.$$

In combination with the classical Gauß theory, the Green's function approach allows to directly relate the vectorial field at altitude R to the vectorial field at altitude r , without need of the scalar Potential U , via

$$\mathbf{B}(\mathbf{x}) = \int_{\Omega} \mathbf{K}_{\text{up}}(\mathbf{x}, R\eta) \mathbf{B}(R\eta) d\omega(\eta), \quad |\mathbf{x}| = r > R,$$

with an explicitly given tensorial kernel \mathbf{K}_{up} . An analogous relation holds true for downward continuation, with a slightly different convolution kernel (although not applicable to crustal field modeling since it requires the field to be source free inside the whole earth).

5. Advantages of Locally Supported Kernels

Even though the local support of the kernels is at the price of losing the physical multipole interpretation, it has the advantage of circumventing the necessity of equidistributed data that is naturally arising in connection with spherical harmonics. A "zooming-in" technique allows to simply approximate regions of higher data density up to higher scales (i.e., an improved resolution of local features) without any deterioration outside the examined region. However, the major advantage for ionospheric modeling from satellite

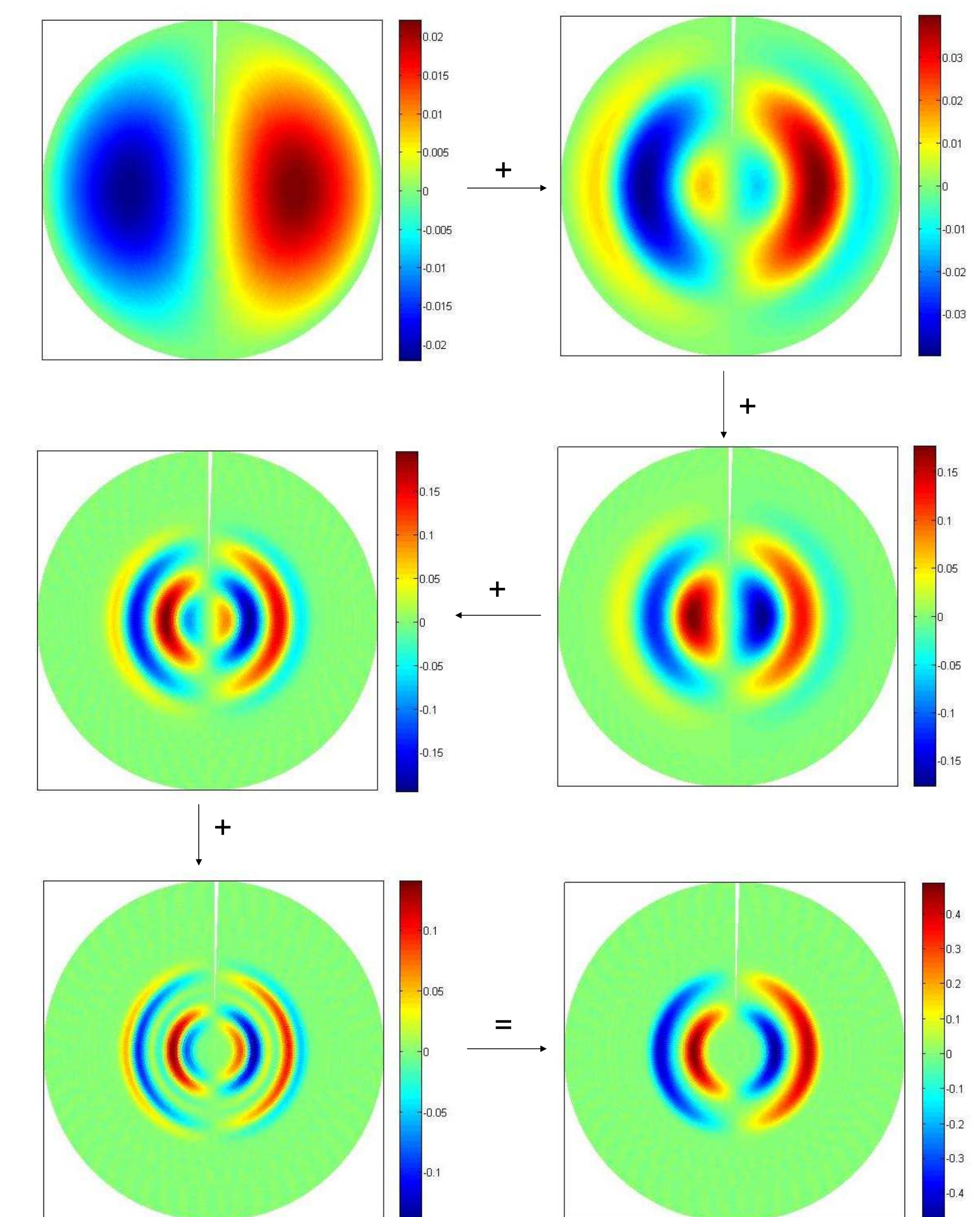


Fig. 2: Reconstructed current at scale 1 (top left) and scale 9 (bottom right) with detail information at scale 2, 4, 6 and 8 (top right to bottom left)

data is the capability of handling large amounts of data. The local support of the kernels limits the required integration to a small scale dependent region. Considering that in the case of radial currents also the scaling kernels have local support, this leads to very fast approximation algorithm.

References

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