A MINIMAL FRAMEWORK FOR DATA ASSIMILATION OF SATELLITE OBSERVATIONS

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Abstract

In order to create maps of the magnetic field and secular variation at the core surface from surface or satellite observations, a method of choosing from the large space of possible solutions must be implemented. The problem is usually attacked by using a two-dimensional representation of the magnetic field on the core surface and adding a regularisation based on some arbitrary function of the field on the core surface, commonly horizontal gradients. Another approach is possible, taking into account both the induction equation and a minimal description of core dynamics. This leads naturally to a 3-D description of the target magnetic field in the entire core volume. One can then use a physical quantity, such as the total Ohmic dissipation in the entire core, as a target to be minimised. The dynamics that are added must encompass Coriolis forces, pressure and Lorentz forces, the so-called magnetostrophic balance. We report here on how this procedure works on static problems, on how it can be posed for instantaneous problems where the rate-of-change is known, and how it might work in principle for a full evolution problem.

1. INTRODUCTION

As was the case at the time of the Magsat launch in 1979, the preparations for the launch of the Swarm mission have contributed, at least in part, to a new impetus in the area of geomagnetic data analysis. Although it is clear that, in the area of magnetometry, the constellation design will lead, for example, to enhanced recovery of crustal fields and possibly new information on mantle electrical conductivity, it is also the case that the recovery of the main field and its secular variation will be enhanced; reliable estimates of the rate-of-change of the field will be delivered to higher spherical harmonic degrees than ever before. It is the main field and its evolution in time that is the focus of the following discussion.

There are two issues that warrant some discussion. The first is that traditional methods of data analysis have concentrated on the problem of estimating the main field and possibly the secular variation and secular acceleration either at the Earth's surface, or at the core-mantle boundary. In the latter problem it is essential to provide some *a priori* information about the types of solutions that are deemed acceptable, as this ill-posed problem suffers from massive amplification of small scale fields. A second area of interest is to attempt to recover the velocity field at the surface of the core, usually by invoking the frozen-flux approximation and also a constraint on the type of motion thought to be likely based on a consideration of the fluid dynamics at the top of the core. Whilst these two procedures of field modelling followed by velocity inversion have been remarkably successful, their combination is suboptimal in the sense that additional information and biasing is added at every stage, in the form of some type of regularisation. This regularisation is invoked in order to find solutions that are smooth and "simple" in a way defined by the norm that is chosen. One can imagine that a method that does not proceed in two steps with injection of smoothing at both steps might be superior.

There is a second issue that may well be brought to the fore by the fact that the resolution of the secular variation recovered by Swarm will be higher than previously, potentially up to spherical harmonic degree 14. As the small scales diffuse faster than large scales, the likelihood of observing magnetic diffusion increases as the resolution increases and the adoption of the frozen-flux hypothesis, used in virtually all recoveries of the velocity field, becomes more suspect. Thus it is of interest to formulate a theory in which diffusion can be accommodated if necessary.

2. THEORY

Our basic viewpoint is that the data concerning the field evolution on the surface of the core should be used within a consistent model of core dynamics. The boundary values provide constraints that must be met by an evolving 3-D model of the interior magnetic field. In our reduced version of the full dynamics (in which we temporarily ignore buoyancy), an initial magnetic field model is sufficient to predict the entire future evolution of the field. Thus the problem we seek to solve is an *initial value problem* where the interior 3-D magnetic field $\mathbf{B}$ at some initial time t_0 is the control variable being sought.

Our starting point for the analysis of the dynamics of the core is based on the classic paper of Taylor [3]. Like him, we envisage a magnetostrophic equilibrium in the core as a likely scenario, in which the Coriolis force, pressure, buoyancy and Lorentz forces all play a role. We review the salient point of Taylor's theory and discuss the implications for a data assimilation strategy. We make use of two coordinate systems, one being the spherical polar coordinates (r, θ, ϕ) and the other being the cylindrical coordinates (s, ϕ, z). Of particular importance is the cylindrical radius s.

When inertial terms (including the nonlinear advection term), and the viscous forces are omitted from the equations of motion in the core, we find the following equation for the fluid velocity:

$$2\rho\Omega \wedge \mathbf{u} = -\nabla p + \mu_0^{-1} \nabla \wedge \mathbf{B} \wedge \mathbf{B} + \rho' g \hat{\mathbf{r}} \tag{1}$$

where $\mathbf{u}$ is velocity, ρ density, ρ' density perturbation from the background state, $\mathbf{B}$ is the magnetic field, Ω is the rotation vector, g is gravity and p is pressure. The omission of inertia in equation (1) is based on the estimated smallness of the Rossby number in the core: it is likely $O(10^{-6})$ when a lengthscale comparable to the size of the core is taken. Even smaller is the Ekman number, $O(10^{-15})$ when molecular viscosities are taken, which means that viscosity can and should be neglected in the bulk of the core.

A repercussion of this magnetostrophic equilibrium, pointed out by Taylor, is that in order for (1) to have solution, a particular constraint must be met, now known as Taylor's constraint:

$$T(s) = \int_{C(s)} [\nabla \wedge \mathbf{B} \wedge \mathbf{B}]_\phi = 0 \quad \forall s \tag{2}$$

where $[]_\phi$ signifies the ϕ component and $C(s)$ is any cylinder coaxial with the rotation axis. Provided this constraint is met then Taylor showed that (1) has a solution when $\mathbf{B}$ and ρ' are given which is unique up to a given unknown azimuthal flow, the so-called geostrophic flow, usually termed $u_G(s)\hat{\boldsymbol{\phi}}$. In other words, when the magnetic field and buoyancy are known one can uniquely find the flow driven by their combination, except for a particular ingredient of the flow that only depends on the cylindrical coordinate s. Omitting buoyancy, the magnetic component of the flow $\mathbf{u}_M$ is determined by

$$\frac{d\mathbf{u}_M}{dz} = (2\rho\Omega\mu_0)^{-1} \nabla \wedge [\nabla \wedge \mathbf{B} \wedge \mathbf{B}] \tag{3}$$

Taylor went on to show much more: when one demands that (2) is met not only at one instance in time but at all times, it is possible to also discover the geostrophic flow $u_G(s)\hat{\boldsymbol{\phi}}$. He argued for the constraint to be always met requires

$$\frac{d}{dt} T(s) = 0 \quad \forall s \tag{4}$$

This leads to a second order equation for u_G that can be solved at every instant in time, so that the field remains in a Taylor state. If we write $u_G = r \sin\theta \omega = s\omega$ then the governing equation for ω, the angular velocity, is of the form

$$\alpha(s) \frac{\partial^2}{\partial s^2} \omega(s) + \beta(s) \frac{\partial}{\partial s} \omega(s) = G \tag{5}$$

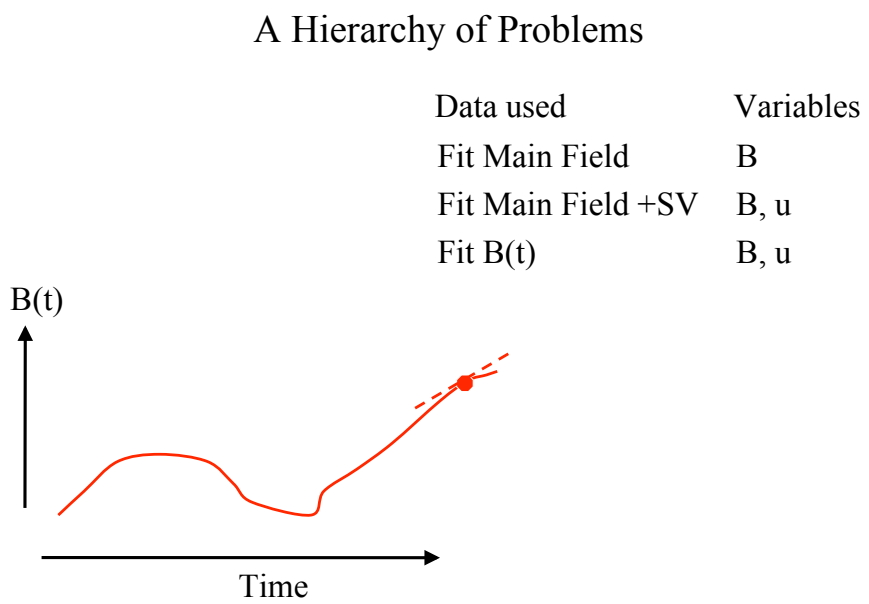

Figure 2.1. Evolution of a boundary value of the magnetic field in the core, such as the l, m component of B_r. Either the value at one instant in time is given (e.g. by the large dot — problem I), the value and its rate-of-change is known (e.g. by the dot and the dashed curve — problem II), or the full evolution is known (problem III). In problems II and III the flow $\mathbf{u}$ *is involved, but this itself is also specified by the magnetic field* $\mathbf{B}$ *through equation (3).*

where $\alpha(s)$ and $\beta(s)$ are known functions of $\mathbf{B}$ and G is a known right hand side depending on $\mathbf{B}$ and $\mathbf{u}_M$. Explicitly,

$$\alpha(s) = \int_{C(s)} sB_s^2 \, dz \, d\phi \tag{6}$$

and

$$\beta(s) = \int_{C(s)} s\mathbf{B} \cdot \nabla B_s + 2B_s^2 \, dz \, d\phi \tag{7}$$

Knowing the instantaneous flow velocity is sufficient to determine the field evolution in time through the induction equation:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla(\mathbf{u} \wedge \mathbf{B}) + \eta \nabla^2 \mathbf{B} \tag{8}$$

where η is the diffusivity of the core (with value of order $1\mathrm{m}^2\mathrm{s}^{-1}$). One can see that these two equations are sufficient to take an initial state $\mathbf{B}_0 = \mathbf{B}(t_0, \mathbf{r})$ and predict an entire evolution of the field for all times $t > t_0$. This is the so-called "direct" or forward problem solved by Taylor.

Our goal in data analysis is to infer what we can about the interior state of the core. Thus a goal would be to determine $\mathbf{B}_0(\mathbf{r})$ that is in accordance with the observations. We concentrate for simplicity on the whole-sphere problem when the inner core is neglected. The presence of the inner core adds considerable complexity to the problem. We can identify a hierarchy of 3 different problems that can be tackled, where we demand agreement with the dynamics and also the observations. These papers draw on the results of Livermore *et al* [1,2], hereinafter termed papers I and II respectively. Figure 1 illustrates schematically the 3 different problems. The field quantity evolving in time is arbitrary; it may be a particular Gauss coefficient or any other quantity. The point is that it is entirely governed by its starting value. In order to represent the field in the interior we make use of the classic Mie representation of solenoidal fields in the form of a poloidal and toroidal expansion:

$$\mathbf{B} = \nabla \wedge \nabla \wedge (S\hat{\mathbf{r}}) + \nabla \wedge (T\hat{\mathbf{r}}) \tag{9}$$

where $S(r, \theta, \phi) = S(\mathbf{r})$ and $T(\mathbf{r})$ are poloidal and toroidal scalars respectively.

2.1. Level 1: Main field known

When observations of B_r are known everywhere on the core surface at an instance in time, supplied by the Gauss coefficients $\{g_l^m; h_l^m\}$ to a certain maximum spherical harmonic degree L, we can find an interior field model that is simultaneously in agreement with (2) and the observations on the boundary. This problem has been examined in papers I and II, where it is shown how interior field models can be found that satisfy Taylor's constraint. In order to choose a particular solution from the large class of Taylor states possible, a particular criterion needs to be adopted to measure "simplicity". Amongst the different measures of complexity presented we will highlight one, namely the Ohmic heating associated with a particular field

$$\Phi = \frac{1}{\mu_0^2 \sigma} \int (\nabla \wedge \mathbf{B})^2 \, dV \tag{10}$$

where μ_0 is the permeability of free space and σ the electrical conductivity of the core. This quantity is rather appealing since it can be related to the heat flow in the system. Another quantity of interest, easily calculated but perhaps not as intuitively obvious, is the rate of entropy increase

$$\Phi = \frac{1}{\mu_0^2 \sigma} \int (\nabla \wedge \mathbf{B})^2 / T \, dV \tag{11}$$

where T is absolute temperature, presumably well approximated by the adiabatic temperature profile in the core.

This subproblem corresponds to the knowledge of one instantaneous point on Figure 1, represented by the dot. Although in a shell there are purely poloidal solutions to this problem (Paper I), we have been unable to find purely poloidal solutions in a sphere that satisfy all the required boundary and regularity conditions; however if one is willing to drop the fact that the field must match to an insulating solution in the exterior of the core, it is possible to find solutions. Solutions exhibited in Paper II show how a combination of toroidal and poloidal fields can work together to minimise the Ohmic dissipation, while also satisfying Taylor's constraint and the given $\{g_l^m; h_l^m\}$.

2.2. Level 2: Main field and rate-of-change known

In this problem both the main field is known (as in the previous problem) and also its rate of change, shown on Figure 1 as the dashed sloping line. We have two sets of constraints we must obey (supplied by the Gauss coefficients $\{g_l^m; h_l^m\}$ and their rates-of-change $\{\dot{g}_l^m; \dot{h}_l^m\}$), and the rate of change is clearly sensitive to the flow that occurs in the core through (8). Let us examine how this problem can be posed, recalling that as always it is the interior field $\mathbf{B}(\mathbf{r})$ that is sought.

The magnetostrophic component of the flow is determined entirely by the magnetic field $\mathbf{B}$ through the equation (3), and indeed it is possible to write its solution explicitly in terms of $\mathbf{B}$ in the form

$$u_i = E_{ijk} b_j b_k \tag{12}$$

where E_{ijk} is a rank 3 tensor and u_i and b_i are coefficients in a spectral expansion of $\mathbf{u}$ and $\mathbf{B}$, which will remain arbitrary at this point, but might, for example, be in spherical harmonics in angular variables and in Worland polynomials in radius. The difficulty might seem to be how to determine the geostrophic component of the flow, governed by the second order differential equation, since the coefficients of the differential equation depend on $\mathbf{B}$ which we do not yet know. There is a resolution to this u_G problem, namely that the geostrophic flow is actually completely determined by the imposed secular variation data on the CMB, since it is z-invariant.

We write the induction equation as

$$\dot{\mathbf{B}} = \nabla \wedge (\mathbf{u}_M \wedge \mathbf{B}) + \nabla \wedge (u_G \hat{\boldsymbol{\phi}} \wedge \mathbf{B}) + \eta \nabla^2 \mathbf{B} \tag{13}$$

where $\mathbf{u}_M$ and $u_G \hat{\boldsymbol{\phi}}$ are the magnetic and geostrophic components of flow. The magnetic component $\mathbf{u}_M$ is determined by (12).

The radial induction equation at the core-mantle boundary reads

$$\dot{B}_r - \eta[\nabla^2\mathbf{B}]_r + \nabla_h \cdot (\mathbf{u}_M B_r) = -u_G \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}B_r = -\omega(s)\partial_\phi B_r \tag{14}$$

where $\partial_\phi = \partial/\partial\phi$. All the pieces on the left hand side are known, in the sense that they can be explicitly written in terms of the unknown poloidal magnetic field scalar S evaluated at $r = 1$. For example, the diffusive term can be written

$$\eta[\nabla^2\mathbf{B}]_r = -\eta\nabla^2\nabla_h^2 S \tag{15}$$

where ∇_h^2 is the horizontal Laplacian.

In almost all circumstances this determines u_G completely, unless B_r is axisymmetric at all longitudes for some θ. It seems logical that the $\partial_\phi B_r$ term be given by the observations, which we'll denote with a tilde, rather than a prediction from the model of S, since this will greatly reduce the nonlinearity of the resulting equations.

Thus in a space-domain schematic of the equations,

$$\omega(s) = -(\partial_\phi \tilde{B}_r)^{-1}[\dot{\tilde{B}}_r - [\eta\nabla^2\mathbf{B}]_r + \nabla_h \cdot (\mathbf{u}_M \tilde{B}_r)] \tag{16}$$

Notice that all the terms can be written in terms of the unknown (as yet) poloidal vector, evaluated at $r = 1$. Also all s-derivatives can be similarly written, because $\partial_s = (\cos\theta)^{-1}\partial_\theta$. We shall require ω'' and ω' in order that we can put it into Taylor's differential equation, and this just depends on θ derivatives of the above equation.

When we insert the expression for u_G into Taylor's differential equation we get a condition that must be obeyed by the magnetic field for the Taylor state to continue to exist. This condition depends on many different polynomial degrees of the magnetic field vector, coupled by different rank tensors. Nevertheless everything is now ready to pass to an optimization routine.

2.3. Level 3: Time evolution known

In this scenario the boundary values are known at a series of points in time. This is truly the data assimilation problem well-known in meteorology and oceanography. The technique has come to the fore in geomagnetism, having been first considered by the collaboration of NASA (GSFC) and University of Maryland, Baltimore County involving Weijia Kuang and Andrew Tangborn and others. Recently the Grenoble group have implemented variational data assimilation in a reduced-dynamics model based on quasi-geostrophy.

Our intention is to use data assimilation via the adjoint method on time series of the Gauss coefficients from the recent past and also the last few hunded years to derive the interior field $\mathbf{B}_0$. We will report on these activities in the future.

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